# REMARKS ON SQUARE FUNCTIONS IN THE LITTLEWOOD-PALEY THEORY

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We prove that certain square function operators in the Littlewood-Paley theory defined by the kernels without any regularity are bounded on $L^p_w$, $1 < p < \infty$, $w \in A_p$ (the weights of Muckenhoupt). Then, we give some applications to the Carleson measures on the upper half space.

## 1. Introduction

In this note we shall prove the weighted $L^p$-estimates for the Littlewood-Paley type square functions arising from kernels satisfying only size and cancellation conditions. Suppose that $\psi \in L^1(\mathbf{R}^n)$ satisfies

$$\int_{\mathbf{R}^n} \psi(x)\, dx = 0. \tag{1.1}$$

We consider a square function of Littlewood-Paley type

$$S(f)(x) = S_\psi(f)(x) = \left( \int_0^\infty |\psi_t \star f(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where $\psi_t(x) = t^{-n}\psi(t^{-1}x)$.

If $\psi$ satisfies, in addition to (1.1),

$$|\psi(x)| \le c(1+|x|)^{-n-\epsilon} \qquad \text{for some} \quad \epsilon > 0 \tag{1.2}$$

$$\int_{\mathbf{R}^n} |\psi(x-y) - \psi(x)|\, dx \le c|y|^\epsilon \qquad \text{for some} \quad \epsilon > 0, \tag{1.3}$$

then it is known that the operator $S$ is bounded on $L^p(\mathbf{R}^n)$ for all $p \in (1, \infty)$ (see Benedek, Calderón and Panzone [1]). Well-known examples are as follows.

**Example 1.** Let $P_t(x)$ be the Poisson kernel for the upper half space $\mathbf{R}^n \times (0, \infty)$:

$$P_t(x) = c_n \frac{t}{(|x|^2 + t^2)^{(n+1)/2}}.$$

Put

$$\psi(x) = \left( \frac{\partial}{\partial t} P_t(x) \right)_{t=1}.$$

Then, $S_\psi(f)$ is the Littlewood-Paley $g$ function.

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**Example 2.** Consider the Haar function $\psi$ on $\mathbf{R}$ :

$$\psi(x) = \chi_{[-1,0]}(x) - \chi_{[0,1]}(x),$$

where $\chi_E$ denotes the characteristic function of a set $E$. Then, $S_\psi(f)$ is the Marcinkiewicz integral

$$\mu(f)(x) = \left( \int_0^\infty |F(x+t) + F(x-t) - 2F(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where $F(x) = \int_0^x f(y)\, dy$.

In this note, we shall prove that the $L^p$-boundedness of $S$ still holds without the assumption (1.3); the conditions (1.1) and (1.2) only are sufficient. This has been already known for the $L^2$-case (see Coifman and Meyer [3, p. 148], and also Journé [7, pp. 81-82] for a proof).

To state our result more precisely, we consider the least non-increasing radial majorant of $\psi$

$$h_\psi(|x|) = \sup_{|y| \geq |x|} |\psi(y)|.$$

We also need to consider two seminorms

$$B_\epsilon(\psi) = \int_{|x|>1} |\psi(x)|\, |x|^\epsilon \, dx \qquad \text{for} \quad \epsilon > 0,$$

$$D_u(\psi) = \left( \int_{|x|<1} |\psi(x)|^u \, dx \right)^{1/u} \qquad \text{for} \quad u > 1.$$

We shall prove the following result.

**Theorem 1.** *Put $H_\psi(x) = h_\psi(|x|)$. If $\psi \in L^1(\mathbf{R}^n)$ satisfies* (1.1) *and*

(1) $B_\epsilon(\psi) < \infty$ *for some $\epsilon > 0$ ;*
(2) $D_u(\psi) < \infty$ *for some $u > 1$ ;*
(3) $H_\psi \in L^1(\mathbf{R}^n)$ *;*

*then the operator $S_\psi$ is bounded on $L^p_w$ :*

$$\|S_\psi(f)\|_{L^p_w} \leq C_{p,w} \|f\|_{L^p_w}$$

*for all $p \in (1, \infty)$ and $w \in A_p$, where $A_p$ denotes the weight class of Muckenhoupt (see* [6,7]*), and*

$$\|f\|_{L^p_w} = \|f\|_{L^p(w)} = \left( \int_{\mathbf{R}^n} |f(x)|^p w(x)\, dx \right)^{1/p}.$$

In fact, we shall prove a more general result.

**Theorem 2.** *Suppose that $\psi \in L^1(\mathbf{R}^n)$ satisfies* (1.1) *and*

(1) $B_\epsilon(\psi) < \infty$ *for some $\epsilon > 0$ ;*
(2) $D_u(\psi) < \infty$ *for some $u > 1$ ;*
(3) $|\psi(x)| \le h(|x|)\Omega(x')$ $(x' = |x|^{-1}x)$ *for some non-negative functions $h$ and $\Omega$ such that*
(a) *$h(r)$ is non-increasing for $r \in (0, \infty)$ ;*
(b) *if $H(x) = h(|x|)$, $H \in L^1(\mathbf{R}^n)$ ;*
(c) *$\Omega \in L^q(S^{n-1})$ for some $q$, $2 \le q \le \infty$.*

*Then, the operator $S_\psi$ is bounded on $L^p_w$ for $p > q'$ and $w \in A_{p/q'}$, where $q'$ denotes the conjugate exponent of $q$.*

When $\psi$ is compactly supported, we have another formulation, which is not included in Theorem 2.

**Theorem 3.** *Suppose that $\psi \in L^1(\mathbf{R}^n)$ satisfies* (1.1) *and*

(1) *$\psi$ is compactly supported ;*
(2) *$\psi \in L^q(\mathbf{R}^n)$ for some $q \ge 2$.*

*Then $S_\psi : L^p_w \to L^p_w$ for $p > q'$ and $w \in A_{p/q'}$.*

These results will be derived from more abstract ones. Let $\psi \in L^1(\mathbf{R}^n)$ satisfy (1.1). We also assume the following :

(1) There exists $\epsilon \in (0, 1)$ such that

$$\int_1^2 |\hat{\psi}(t\xi)|^2\, dt \le c \min\left(|\xi|^{\epsilon}, |\xi|^{-\epsilon}\right) \qquad \text{for all} \quad \xi \in \mathbf{R}^n \ , \tag{1.4}$$

where $\hat{\psi}$ denotes the Fourier transform

$$\hat{\psi}(\xi) = \int \psi(x) e^{-2\pi i \langle x, \xi\rangle}\, dx, \qquad \langle x, \xi\rangle = \sum_{j=1}^n x_j \xi_j \quad (\text{the inner product in } \mathbf{R}^n).$$

(2) Let $1 \le s \le 2$. For all $w \in A_s$, we have

$$\sup_{k \in \mathbf{Z}} \int_{\mathbf{R}^n} \int_1^2 \left|\psi_{t2^k} \star f(x)\right|^2\, dt\, w(x)\, dx \le C_w \|f\|^2_{L^2_w} \qquad \text{for all} \quad f \in \mathcal{S}(\mathbf{R}^n) \ , \tag{1.5}$$

where $\mathbf{Z}$ denotes the integer group and $\mathcal{S}(\mathbf{R}^n)$ the Schwartz space.

Under these assumptions the following holds.

**Proposition 1.** *For $p > 2/s$ and $w \in A_{ps/2}$, the operator $S_\psi$ is bounded on $L^p_w$.*

This will be used to prove the next result.

**Proposition 2.** *Put*

$$J_\epsilon(\psi) = \sup_{|\xi|=1} \iint_{\mathbf{R}^n \times \mathbf{R}^n} |\psi(x)\psi(y)|\, |\langle \xi, x - y\rangle|^{-\epsilon}\, dx\, dy \qquad \textit{for} \quad \epsilon \in (0, 1].$$

*Let $\psi \in L^1$ satisfy* (1.1) *and* (1.5)*. Then if $B_\epsilon(\psi) < \infty$ and $J_\epsilon(\psi) < \infty$ for some $\epsilon \in (0, 1]$, the operator $S_\psi$ is bounded on $L^p_w$ for $p > 2/s$ and $w \in A_{ps/2}$.*

In §2, we shall prove Proposition 1 by the method of the proof of Duoandikoetxea and Rubio de Francia [5, Corollary 4.2] and then Proposition 2 by using Proposition

1. Proposition 2 will be applied to prove Theorems 2 and 3 in §3. Finally, in §4, we shall give some applications of Theorem 1 to generalized Marcinkiewicz integrals and the Carleson measures on the upper half space $\mathbf{R}^n \times (0,\infty)$.

To conclude this section, we state a result for the $L^2$-case, from which the result of Coifman-Meyer mentioned above immediately follows, and an idea of the proof will be applied later too (see the proof of Lemma 2).

**Proposition 3.** *Suppose that* $\psi \in L^1$ *satisfies* (1.1). *Let*

$$L(\psi) = \sup_{|\xi|=1} \iint_{\mathbf{R}^n \times \mathbf{R}^n} |\psi(x)\psi(y)| \left|\log|\langle \xi, x-y\rangle|\right| \, dx\, dy.$$

*Then, if* $L(\psi) < \infty$, *the operator* $S_\psi$ *is bounded on* $L^2$.

*Proof.* It is sufficient to show that

$$\sup_{|\xi|=1} \int_0^\infty \left|\hat{\psi}(t\xi)\right|^2 \frac{dt}{t} < \infty.$$

We write

$$|\hat{\psi}(t\xi)|^2 = \hat{\psi}(t\xi)\overline{\hat{\psi}(t\xi)} = \iint_{\mathbf{R}^n \times \mathbf{R}^n} \psi(x)\overline{\psi(y)} e^{-2\pi i t\langle \xi, x-y\rangle}\, dx\, dy,$$

and so

$$\int_0^\infty \left|\hat{\psi}(t\xi)\right|^2 \frac{dt}{t} = \lim_{N\to\infty, \epsilon\to 0} \iint \psi(x)\overline{\psi(y)} \left(\int_\epsilon^N e^{-2\pi i t\langle \xi, x-y\rangle} \frac{dt}{t}\right) dx\, dy.$$

Note that

$$\int_\epsilon^N \left(e^{-2\pi i t\langle \xi, x-y\rangle} - \cos(2\pi t)\right) \frac{dt}{t} \to -\log|\langle \xi, x-y\rangle| - i\frac{\pi}{2}\operatorname{sgn}\langle \xi, x-y\rangle$$

as $N \to \infty$ and $\epsilon \to 0$, and the integral is bounded, uniformly in $\epsilon$ and $N$, by

$$c\left(1 + \left|\log|\langle \xi, x-y\rangle|\right|\right).$$

Thus, using (1.1) and the dominated convergence theorem, we get

$$\int_0^\infty \left|\hat{\psi}(t\xi)\right|^2 \frac{dt}{t} = \iint \psi(x)\overline{\psi(y)} \left(-\log|\langle \xi, x-y\rangle| - i\frac{\pi}{2}\operatorname{sgn}\langle \xi, x-y\rangle\right) dx\, dy.$$

This immediately implies the conclusion.

*Remark.* In the one-dimensional case, it is easy to see that if

$$\int |\psi(x)|\log(2+|x|)\, dx < \infty \qquad \text{and} \qquad \int |\psi(x)|\log(2+|\psi(x)|)\, dx < \infty,$$

then $L(\psi) < \infty$, and so $S_\psi : L^2 \to L^2$.

## 2. Proofs of Propositions 1 and 2

We use a Littlewood-Paley decomposition. Let $f \in \mathcal{S}(\mathbf{R}^n)$, and define

$$\widehat{\Delta_j(f)}(\xi) = \Psi(2^j\xi)\hat{f}(\xi) \qquad \text{for} \quad j \in \mathbf{Z},$$

where $\Psi \in C^\infty$ is supported in $\{1/2 \le |\xi| \le 2\}$ and satisfies

$$\sum_{j\in\mathbf{Z}} \Psi(2^j\xi) = 1 \qquad \text{for} \quad \xi \neq 0.$$

Decompose

$$f \star \psi_t(x) = \sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} \Delta_{j+k}(f\star\psi_t)(x)\chi_{[2^k,2^{k+1})}(t) = \sum_{j\in\mathbf{Z}} F_j(x,t), \quad \text{say,}$$

and define

$$T_j(f)(x) = \left(\int_0^\infty |F_j(x,t)|^2\,\frac{dt}{t}\right)^{1/2}.$$

Then

$$S(f)(x) \le \sum_{j\in\mathbf{Z}} T_j(f)(x).$$

Put $E_j = \{2^{-1-j} \le |\xi| \le 2^{1-j}\}$. Then by the Plancherel theorem and (1.4) we have

$$\begin{aligned}
\|T_j(f)\|_2^2 &= \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n}\int_{2^k}^{2^{k+1}} |\Delta_{j+k}\left(f\star\psi_t\right)(x)|^2\,\frac{dt}{t}\,dx \\
&\le \sum_{k\in\mathbf{Z}} c\int_{E_{j+k}} \left(\int_{2^k}^{2^{k+1}} \left|\hat{\psi}(t\xi)\right|^2\,\frac{dt}{t}\right)\left|\hat{f}(\xi)\right|^2\,d\xi \\
&\le \sum_{k\in\mathbf{Z}} c\int_{E_{j+k}} \min\left(|2^k\xi|^\epsilon, |2^k\xi|^{-\epsilon}\right)\left|\hat{f}(\xi)\right|^2\,d\xi \\
&\le c2^{-\epsilon|j|}\sum_{k\in\mathbf{Z}}\int_{E_{j+k}} \left|\hat{f}(\xi)\right|^2\,d\xi \\
&\le c2^{-\epsilon|j|}\|f\|_2^2,
\end{aligned}$$

where the last inequality holds since the sets $E_j$ are finitely overlapping. (We denote by $\|\cdot\|_p$ the ordinary $L^p$-norm.)

On the other hand, for $w \in A_s$ by (1.5) we see that

$$\begin{aligned}
\|T_j(f)\|_{L^2_w}^2 &= \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n}\int_{2^k}^{2^{k+1}} |\Delta_{j+k}(f)\star\psi_t(x)|^2\,\frac{dt}{t}\,w(x)\,dx \\
&\le \sum_{k\in\mathbf{Z}} c\int_{\mathbf{R}^n} |\Delta_{j+k}(f)(x)|^2\,w(x)\,dx \\
&\le c\|f\|_{L^2_w}^2,
\end{aligned}$$

where the last inequality follows from a well-known Littlewood-Paley inequality for $L^2_w$ since $A_s \subset A_2$.

Interpolating with change of measures between the two estimates above, we get

$$\|T_j(f)\|_{L^2(w^u)} \leq c 2^{-\epsilon(1-u)|j|/2} \|f\|_{L^2(w^u)}$$

for $u \in (0,1)$. If we choose $u$ (close to 1) so that $w^{1/u} \in A_s$, then from this inequality we get

$$\|T_j(f)\|_{L^2_w} \leq c 2^{-\epsilon(1-u)|j|/2} \|f\|_{L^2_w},$$

and so

$$\|S(f)\|_{L^2_w} \leq \sum_{j \in \mathbf{Z}} \|T_j(f)\|_{L^2_w} \leq c\|f\|_{L^2_w}.$$

Thus the extrapolation theorem of Rubio de Francia [8] implies the conclusion.

To derive Proposition 2 from Proposition 1 we prepare the following lemmas.

**Lemma 1.** *If $\psi \in L^1(\mathbf{R}^n)$ satisfies* (1.1) *and $B_\epsilon(\psi) < \infty$ for $\epsilon \in (0,1]$, then*

$$|\hat{\psi}(\xi)| \leq c|\xi|^\epsilon \qquad \textit{for all} \quad \xi \in \mathbf{R}^n.$$

*Proof.* Since $a \leq a^\epsilon$ for $a, \epsilon \in (0,1]$, we see that

$$\begin{aligned}|\hat{\psi}(\xi)| = \left| \int \psi(x) \left( e^{-2\pi i \langle x, \xi \rangle} - 1 \right) dx \right| &\leq c \int |\psi(x)| \min(1, |\langle x, \xi \rangle|) \, dx \\ &\leq c|\xi|^\epsilon \int |\psi(x)||x|^\epsilon \, dx.\end{aligned}$$

This completes the proof.

**Lemma 2.** *If $\psi \in L^1(\mathbf{R}^n)$ and $J_\epsilon(\psi) < \infty$ for $\epsilon \in (0,1]$, then*

$$\int_1^2 |\hat{\psi}(t\xi)|^2 \, dt \leq c|\xi|^{-\epsilon} \qquad \textit{for all} \quad \xi \in \mathbf{R}^n.$$

*Proof.* As in the proof of Proposition 3, we see that

$$\int_1^2 |\hat{\psi}(t\xi)|^2 \, dt = \iint_{\mathbf{R}^n \times \mathbf{R}^n} \psi(x)\overline{\psi(y)} \frac{e^{-4\pi i \langle \xi, x-y \rangle} - e^{-2\pi i \langle \xi, x-y \rangle}}{-2\pi i \langle \xi, x-y \rangle} \, dx \, dy.$$

Thus

$$\begin{aligned}\int_1^2 |\hat{\psi}(t\xi)|^2 \, dt &\leq c \iint_{\mathbf{R}^n \times \mathbf{R}^n} |\psi(x)\psi(y)| \min\left(1, |\langle \xi, x-y \rangle|^{-1}\right) \, dx \, dy \\ &\leq c J_\epsilon(\psi) \, |\xi|^{-\epsilon}.\end{aligned}$$

This completes the proof.

Now, we can see that Proposition 1 implies Proposition 2, since the condition (1.4) follows from Lemmas 1 and 2.

## 3. Proofs of Theorems 2 and 3

To get Theorem 2 from Proposition 2 we need Lemmas 3 and 4 below. First, we give a sufficient condition for $J_\epsilon(\psi) < \infty$.

**Lemma 3.** *Let $h(r)$, $h \geq 0$, be a non-increasing function for $r > 0$ satisfying $H \in L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$, where $H(x) = h(|x|)$, and let $\Omega \in L^v(S^{n-1})$, $v > 1$, $\Omega \geq 0$. Suppose that $F$ is a non-negative function such that*

$$F(x) \leq h(|x|)\Omega(x') \qquad \textit{for} \quad |x| > 1$$

*and $D_u(F) < \infty$ for $u > 1$. Then $J_\epsilon(F) < \infty$ if $\epsilon < \min(1/u', 1/v')$.*

*Proof.* For non-negative functions $f$, $g$ and $\xi \in S^{n-1}$ put

$$L_\epsilon(f, g; \xi) = \iint_{\mathbf{R}^n \times \mathbf{R}^n} f(x)g(y) \left|\langle \xi, x - y\rangle\right|^{-\epsilon} \, dx \, dy.$$

Decompose $F$ as $F = E + G$, where $E(x) = F(x)$ if $|x| < 1$ and $E(x) = 0$ otherwise. Then

$$L_\epsilon(F, F; \xi) = L_\epsilon(E, E; \xi) + 2L_\epsilon(E, G; \xi) + L_\epsilon(G, G; \xi).$$

We show that each of $L_\epsilon(E, E; \xi)$, $L_\epsilon(E, G; \xi)$ and $L_\epsilon(G, G; \xi)$ is bounded by a constant independent of $\xi$ if $\epsilon < \min(1/u', 1/v')$.

First, by Hölder's inequality and a change of variables

$$L_\epsilon(E, E; \xi) \leq \|E\|_u^2 \left( \iint_{|x|<1, |y|<1} |x_1 - y_1|^{-\epsilon u'} \, dx \, dy \right)^{1/u'},$$

where we note that $\|E\|_u = D_u(F)$.

Next, by Hölder's inequality again

$$L_\epsilon(E, G; \xi) \leq \|E\|_u \left( \int_{|x|<1} \left( \int_{\mathbf{R}^n} G(y) \left|x_1 - \langle \xi, y\rangle\right|^{-\epsilon} \, dy \right)^{u'} dx \right)^{1/u'}.$$

For $s > 0$, let

$$I_\epsilon(s) = \int_{S^{n-1}} \left|x_1 - \langle \xi, s\omega\rangle\right|^{-\epsilon} \Omega(\omega) \, d\sigma(\omega)$$

for fixed $x_1$ and $\xi$, where $d\sigma$ denotes the Lebesgue surface measure of $S^{n-1}$ (when $n = 1$, let $\sigma(\{1\}) = \sigma(\{-1\}) = 1$). Then by Hölder's inequality

$$I_\epsilon(s) \leq \left(N_{\epsilon v'}(s)\right)^{1/v'} \|\Omega\|_v,$$

where

$$N_\epsilon(s) = \int_{S^{n-1}} |x_1 - s\omega_1|^{-\epsilon} \, d\sigma(\omega).$$

Thus, using Hölder's inequality,

$$\begin{aligned}\int_{\mathbf{R}^n} G(y)\,|x_1-\langle\xi,y\rangle|^{-\epsilon}\,dy &\le \int_0^\infty h(s)s^{n-1}I_\epsilon(s)\,ds\\ &\le \|\Omega\|_v\int_0^\infty h(s)s^{n-1}\left(N_{\epsilon v'}(s)\right)^{1/v'}\,ds\\ &\le c\|H\|_1^{1/v}\|\Omega\|_v\left(\int_0^\infty h(s)s^{n-1}N_{\epsilon v'}(s)\,ds\right)^{1/v'}\\ &= c\|H\|_1^{1/v}\|\Omega\|_v\left(\int_{\mathbf{R}^n} h(|y|)\,|x_1-y_1|^{-\epsilon v'}\,dy\right)^{1/v'}.\end{aligned}$$

Therefore, the desired estimate for $L_\epsilon(E,G;\xi)$ follows if we show that

$$\sup_{x_1\in\mathbf{R}}\int_{\mathbf{R}^n} h(|y|)\,|x_1-y_1|^{-\epsilon v'}\,dy<\infty. \tag{4.1}$$

To see this, we split the domain of the integration as follows :

$$\begin{aligned}\int_{\mathbf{R}^n} h(|y|)\,|x_1-y_1|^{-\epsilon v'}\,dy &= \int_{|x_1-y_1|<1} h(|y|)\,|x_1-y_1|^{-\epsilon v'}\,dy\\ &\quad+\int_{|x_1-y_1|>1} h(|y|)\,|x_1-y_1|^{-\epsilon v'}\,dy\\ &= I_1+I_2,\quad\text{say.}\end{aligned}$$

Clearly $I_2\le\|H\|_1$. To estimate $I_1$ we may assume that $n\ge 2$; the case $n=1$ can be easily disposed of since $h$ is bounded. We need further splitting of the domain of the integration. We write $y=(y_1,y')$, $y'\in\mathbf{R}^{n-1}$. Then

$$\begin{aligned}I_1 &= \int\limits_{\substack{|x_1-y_1|<1\\ |y'|<1}} h(|y|)\,|x_1-y_1|^{-\epsilon v'}\,dy+\int\limits_{\substack{|x_1-y_1|<1\\ |y'|>1}} h(|y|)\,|x_1-y_1|^{-\epsilon v'}\,dy\\ &= I_3+I_4,\quad\text{say.}\end{aligned}$$

It is easy to see that

$$I_3\le\|H\|_\infty\int_{|y|<2}|y_1|^{-\epsilon v'}\,dy<\infty.$$

Next, since $h(|y|)\le h(|y'|)$,

$$\begin{aligned}I_4 &\le \int_{|y_1|<1}|y_1|^{-\epsilon v'}\,dy_1\int_{|y'|>1}h(|y'|)\,dy'\\ &\le c\int_{|y_1|<1}|y_1|^{-\epsilon v'}\,dy_1\int_{|y|>1}h(|y|)\,dy<\infty.\end{aligned}$$

It remains to estimate $L_\epsilon(G,G;\xi)$. Note that

$$L_\epsilon(G,G;\xi)\le\int_0^\infty\int_0^\infty h(r)h(s)r^{n-1}s^{n-1}I_\epsilon(r,s)\,dr\,ds, \tag{4.2}$$

where

$$I_\epsilon(r,s) = \iint_{S^{n-1}\times S^{n-1}} |\langle \xi, r\theta - s\omega\rangle|^{-\epsilon}\,\Omega(\theta)\Omega(\omega)\,d\sigma(\theta)\,d\sigma(\omega).$$

By Hölder's inequality

$$I_\epsilon(r,s) \le \left(N_{\epsilon v'}(r,s)\right)^{1/v'} \|\Omega\|_v^2, \tag{4.3}$$

where

$$N_\epsilon(r,s) = \iint_{S^{n-1}\times S^{n-1}} |r\theta_1 - s\omega_1|^{-\epsilon}\, d\sigma(\theta)\,d\sigma(\omega).$$

Using the estimate (4.3) in (4.2) and then applying Hölder's inequality, we see that

$$\begin{aligned} L_\epsilon(G,G;\xi) &\le c\|H\|_1^{2/v}\|\Omega\|_v^2 \left(\int_0^\infty\int_0^\infty N_{\epsilon v'}(r,s)h(r)h(s)r^{n-1}s^{n-1}\,dr\,ds\right)^{1/v'} \\ &= c\|H\|_1^{2/v}\|\Omega\|_v^2 \left(\iint_{\mathbf{R}^n\times\mathbf{R}^n} h(|x|)h(|y|)\,|x_1-y_1|^{-\epsilon v'}\,dx\,dy\right)^{1/v'}. \end{aligned}$$

Therefore, the desired estimates follows again from (4.1). This completes the proof.

For a non-negative function $\Omega$ on $S^{n-1}$ we define a non-isotropic Hardy-Littlewood maximal function

$$M_\Omega(f)(x) = \sup_{r>0} r^{-n}\int_{|y|<r} |f(x-y)|\Omega(|y|^{-1}y)\,dy.$$

To prove Theorem 2 we also need the following (see Duoandikoetxea [4]).

**Lemma 4.** *If $\Omega \in L^q(S^{n-1})$, $q \ge 2$, and $w \in A_{2/q'}$, then $M_\Omega$ is bounded on $L^2_w$.*

Now we can prove Theorem 2. As in Stein [10, pp. 63-64], we can show that

$$\sup_{t>0} |\psi_t \star f(x)| \le c\, M_\Omega(f).$$

So, by Lemma 4 we see that the condition (1.5) holds for $\psi$ of Theorem 2 with $s = 2/q'$.

Next, applying Lemma 3, we see that $J_\epsilon(\psi) < \infty$ for $\epsilon < \min(1/u', 1/q')$ (note that $h(r)$ of Theorem 2 (3) is bounded for $r \ge 1$). Combining these facts with the assumption in Theorem 2 (1), we can apply Proposition 2 to reach the conclusion.

Finally, we give the proof of Theorem 3. Clearly $B_1(\psi) < \infty$, and $J_{1/(2q')}(\psi) < \infty$ by applying Lemma 3 suitably. Therefore, the conclusion follows from Proposition 2 if we show that the condition (1.5) holds with $s = 2/q'$. But, for $q > 2$ this is a consequence of the inequality

$$\sup_{t>0} |\psi_t \star f(x)| \le c\, M(|f|^{q'})^{1/q'},$$

where $M$ denotes the Hardy-Littlewood maximal operator. (This inequality is easily seen by Hölder's inequality.)

To prove the condition (1.5) when $q = 2$ and $w \in A_1$, we may assume that $\psi$ is supported in $\{|x| < 1\}$. Then by Schwarz's inequality

$$|\psi_t \star f(x)|^2 \le t^{-n}\|\psi\|_2^2 \int_{|y|<t} |f(x-y)|^2\,dy.$$

Integrating with the measure $w(x)\,dx$ and using a property of the $A_1$-weight function, we get

$$\begin{aligned}\int |\psi_t \star f(x)|^2\,w(x)\,dx &\le \|\psi\|_2^2 \int |f(y)|^2 t^{-n} \int_{|x-y|<t} w(x)\,dx\,dy \\ &\le C_w\|\psi\|_2^2 \int |f(y)|^2 w(y)\,dy\end{aligned}$$

uniformly in $t$. From this the desired inequality follows.

## 4. Applications

It is to be noted that Theorem 1 can be applied to study the $L^p_w$-boundedness of generalized Marcinkiewicz integrals.

**Corollary 1.** *For $\epsilon > 0$, let*

$$\psi(x) = |x|^{-n+\epsilon}\Omega(x')\chi_{(0,1]}(|x|),$$

*where $\Omega \in L^\infty(S^{n-1})$ and $\int \Omega(x')\,d\sigma(x') = 0$. Define a Marcinkiewicz integral*

$$\mu(f)(x) = \left(\int_0^\infty |\psi_t \star f(x)|^2\,\frac{dt}{t}\right)^{1/2}.$$

*Then, the operator $\mu$ is bounded on $L^p_w$ for all $p \in (1,\infty)$ and $w \in A_p$ :*

$$\|\mu(f)\|_{L^p_w} \le C_{p,w}\|f\|_{L^p_w}.$$

This result, in particular, removes the Lipschitz condition assumed for $\Omega$ in Stein [9, Theorem 1 (2)].

Next, we consider applications to the Carleson measures on the upper half spaces.

**Corollary 2.** *Suppose $\psi \in L^1$ satisfies* (1.1) *and*

$$|\psi(x)| \le c(1+|x|)^{-n-\epsilon} \qquad \text{for some} \quad \epsilon > 0.$$

*Take $b \in BMO$ and $w \in A_2$. Then the measure*

$$d\nu(x,t) = |\psi_t \star b(x)|^2\,\frac{dt}{t}\,w(x)\,dx$$

*on the upper half space $\mathbf{R}^n \times (0,\infty)$ is a Carleson measure with respect to the measure $w(x)\,dx$, that is,*

$$\nu(S(Q)) \le C_w\|b\|_{BMO}^2 \int_Q w(x)\,dx$$

*for all cubes $Q$ in $\mathbf{R}^n$, where*

$$S(Q) = \{(x,t) \in \mathbf{R}^n \times (0,\infty) : x \in Q, 0 < t \le \ell(Q)\},$$

*with $\ell(Q)$ denoting sidelength of $Q$.*

This can be proved by using $L^2_w$-boundedness of the operator $S_\psi$ (see Theorem 1) as in Journé [7, Chap. 6 III, pp. 85–87]. In [7], a similar result has been proved with an additional assumption on the gradient of $\psi$.

Arguing as in [7, Chap. 6 III, p. 87], by Corollary 2 we can get the following.

**Corollary 3.** *Let $\psi$ and $b$ be as in Corollary 2. Suppose $\varphi$ satisfies*

$$|\varphi(x)| \le c(1+|x|)^{-n-\delta}$$

*for $\delta > 0$. Then, the sublinear operator*

$$T_b(f)(x) = \left( \int_0^\infty |\psi_t \star b(x)|^2 \, |\varphi_t \star f(x)|^2 \, \frac{dt}{t} \right)^{1/2}$$

*is bounded on $L^p_w$ for all $p \in (1,\infty)$ and $w \in A_p$ :*

$$\|T_b(f)\|_{L^p_w} \le C_{p,w} \|b\|_{BMO} \|f\|_{L^p_w}.$$

Here again we don't need the assumption on the gradient of $\psi$. See Coifman and Meyer [3, p. 149] for the $L^2$-case.

**Corollary 4.** *Suppose $\eta \in L^1(\mathbf{R}^n)$ satisfies the assumptions of Theorem 1 for $\psi$. Let $\psi$, $\varphi$ and $b$ be as in Corollary 3, and define a paraproduct*

$$\pi_b(f)(x) = \int_0^\infty \eta_t \star ((\psi_t \star b)(\varphi_t \star f))(x) \, \frac{dt}{t}.$$

*Then, the operator $\pi_b$ is bounded on $L^p_w$ for all $p \in (1,\infty)$ and $w \in A_p$ :*

$$\|\pi_b(f)\|_{L^p_w} \le C_{p,w} \|b\|_{BMO} \|f\|_{L^p_w}.$$

*Proof.* Let $g \in L^2(w^{-1})$, $w \in A_2$. Then, since $w^{-1} \in A_2$, by Schwarz's inequality, Theorem 1 and Corollary 3, for $0 < u < v$, we see that

$$\begin{aligned}
&\left| \int \int_u^v \eta_t \star ((\psi_t \star b)(\varphi_t \star f))(x) \, \frac{dt}{t} g(x) \, dx \right| \\
&\le \left( \int \int_u^v |\tilde{\eta}_t \star g(x)|^2 \, \frac{dt}{t} \, w^{-1}(x) \, dx \right)^{1/2} \|T_b(f)\|_{L^2(w)} \\
&\le C_w \|b\|_{BMO} \|g\|_{L^2(w^{-1})} \|f\|_{L^2(w)},
\end{aligned}$$

where $\tilde{\eta}(x) = \eta(-x)$. From this estimate we can see that $\pi_b(f)$ is well-defined (see Christ [2, III, §3]). Taking the supremum over $g$ with $\|g\|_{L^2(w^{-1})} \le 1$, we get the $L^2_w$-boundedness, and so the extrapolation theorem of Rubio de Francia implies the conclusion. This completes the proof.

See Coifman and Meyer [3, p. 149, PROPOSITION 1] for a similar result in the $L^2$-case.

## References

1. A. Benedek, A. P. Calderón and R. Panzone, *Convolution operators on Banach space valued functions*, Proc. Nat. Acad. Sci. U.S.A. **48** (1962), 356–365.
2. M. Christ, *Lectures on singular integral operators*, Regional conference series in mathematics, no. 77, Amer. Math. Soc., 1990.
3. R. R. Coifman and Y. Meyer, *Au delà des opérateurs pseudo-différentiels*, Astérisque no. 57, Soc. Math. France, 1978.

4. J. Duoandikoetxea, *Weighted norm inequalities for homogeneous singular integrals*, Trans. Amer. Math. Soc. **336** (1993), 869–880.
5. J. Duoandikoetxea and J. L. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates*, Invent. Math. **84** (1986), 541–561.
6. J. Garcia-Cuerva and J. L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland, 1985.
7. J.-L. Journé, *Calderón-Zygmund operators, pseudo-differential operators and the Cauchy integral of Calderón*, Lecture Notes in Math. vol. 994, Springer-Verlag, 1983.
8. J.L. Rubio de Francia, *Factorization theory and $A_p$ weights*, Amer. J. Math. **106** (1984), 533-547.
9. E. M. Stein, *On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz*, Trans. Amer. Math. Soc. **88** (1958), 430–466.
10. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, 1970.

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